

# The three different regimes in coulombic friction

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**Abstract.** de Gennes identified three regimes in the phenomenon of the Langevin equation which includes Coulombic friction. Here we extend and precise this phenomenon to a constant external force.

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## 1. Introduction

P-G de Gennes [3] studied the Langevin equation under the influence of a dry friction force modelled by the equation

$$dv = -\frac{1}{2}\Delta \operatorname{sgn}(v)dt + \sqrt{D}dB,$$

the dry friction force with threshold force  $\Delta > 0$ , and  $D > 0$  is the diffusion coefficient. Here  $B$  is the standard Brownian motion, and  $\operatorname{sgn}(v) = 1$  if  $v > 0$ , and  $\operatorname{sgn}(v) = -1$  if  $v < 0$ . Comparing the magnitude of  $\alpha$ ,  $\Delta$  and  $D$  de Gennes [3] identified three different regimes: viscous, partly stuck and stuck.

Later Touchette et al. [7] extended de Gennes work by calculating the time-dependent propagator of the Langevin equation

$$dv = -\frac{1}{2}[\alpha v - a + \Delta \operatorname{sgn}(v)]dt + \sqrt{D}dB, \quad (1)$$

which includes a constant external force  $a \in \mathbb{R}$ .

In this paper, we precise and extend de Gennes's work to the Langevin equation (1) and find again the result of Touchette et al. [7] using the trivariate density of Brownian motion, its local and occupation times.

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## 2. The three different regimes in coulombic friction

If  $v(t)$  is solution of (1), then  $v(\frac{t}{D})$  satisfies the equation

$$dv = -\frac{1}{2D}[\alpha v - a + \Delta \operatorname{sgn}(v)]dt + dB. \quad (2)$$

It follows that for large time  $T$  the PDF of the velocity  $v(\frac{T}{D})$  is approximated by the stationary PDF

$$\frac{1}{Z} \exp \left[ -\frac{1}{\nu} \left( \frac{(v-y)^2}{2\tau} + |v| \right) \right],$$

where

$$Z = \frac{1}{2\nu} \left[ \exp\left(\frac{\tau-2y}{2\nu}\right) G\left(\frac{\tau-y}{\sqrt{\tau\nu}}\right) + \exp\left(\frac{\tau+2y}{2\nu}\right) G\left(\frac{\tau+y}{\sqrt{\tau\nu}}\right) \right]$$

is the partition function i.e. the normalization constant. Here and the sequel

$$G(u) = \frac{1}{\sqrt{2\pi}} \int_u^{+\infty} \exp\left(-\frac{v^2}{2}\right) dv, \quad \nu = \frac{D}{\Delta}, \quad \tau = \frac{\Delta}{\alpha}, \quad y = \frac{a}{\alpha}.$$

We say that the stochastic process  $(V_D : D > 0)$  defined in some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  converges in probability distribution as  $D \rightarrow 0$  to the PDF  $f$  if for each couple  $l < r$  of real numbers

$$\mathbb{P}(l \leq V_D \leq r) \rightarrow \int_l^r f(v) dv, \quad \text{as } D \rightarrow 0.$$

Now we can announce our result.

1) Stuck regime. If  $|a| < \Delta$ , then the velocity  $v(\frac{T}{D}) \rightarrow 0$  as  $D \rightarrow 0$ . More precisely  $\frac{1}{\nu}v(\frac{T}{D})$  converges in distribution as  $D \rightarrow 0$  to the PDF

$$\frac{1-y^2}{2} \exp[-|v|(1-\operatorname{sgn}(v)|y|)].$$

Observe that if the constant force  $a = 0$ , then  $y = 0$  and the limit is

$$\frac{1}{2} \exp(-|v|).$$

2) Partly stuck regime. If  $|a| = \Delta$ , then the velocity  $v(\frac{T}{D}) \rightarrow 0$  as  $D \rightarrow 0$ . More precisely we distinguish two cases.

a) If we consider only the event  $av(\frac{T}{D}) < 0$ , then

$$\frac{1}{\nu}v(\frac{T}{D}) \rightarrow 2 \exp(-2|v|) \mathbf{1}_{[av < 0]} \quad \text{as } D \rightarrow 0.$$

b) If we consider only the event  $av(\frac{T}{D}) > 0$ , then

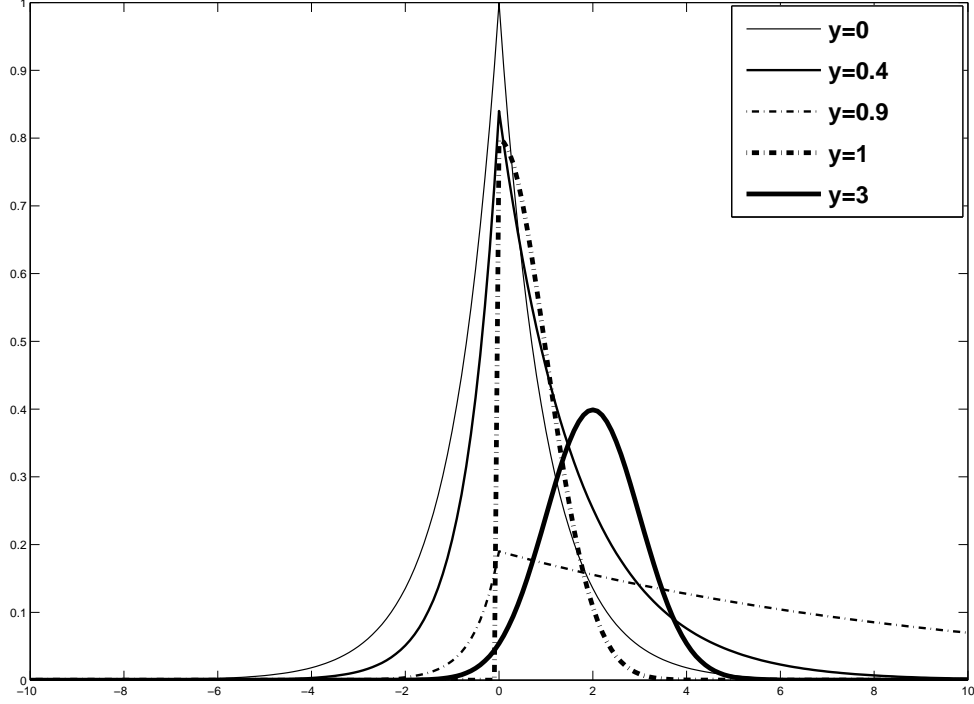
$$\frac{1}{\sqrt{\nu}}v(\frac{T}{D}) \rightarrow \frac{2}{\sqrt{2\pi\tau}} \exp(-\frac{v^2}{2\tau}) \mathbf{1}_{[av > 0]} \quad \text{as } D \rightarrow 0.$$

Moreover the probability of the event  $av(\frac{T}{D}) > 0$  tends to 1 as  $D \rightarrow 0$ . Hence  $\frac{1}{\sqrt{\nu}}v(\frac{T}{D})$  converges to  $\frac{2}{\sqrt{2\pi\tau}} \exp(-\frac{v^2}{2\tau}) \mathbf{1}_{[av > 0]}$ .

3) Viscous regime. If  $|a| > \Delta$  then as  $D \rightarrow 0$  the velocity  $v(\frac{T}{D})$  becomes Gaussian with the mean  $(y - \operatorname{sgn}(y)\tau)$  and the variance  $\nu\tau$ . More precisely, we have

$$\frac{v(\frac{T}{D}) - (y - \operatorname{sgn}(y)\tau)}{\sqrt{\nu}} \rightarrow \frac{1}{\sqrt{2\pi\tau}} \exp(-\frac{v^2}{2\tau}).$$

Observe that the asymptotic mean  $y - \text{sgn}(y)\tau$  is the minimizer of the potential  $v \rightarrow \frac{(v-y)^2}{2\tau} + |v| := U(v)$ .



**Figure 1.** Three scenarios of the stuck regime with  $y = 0, 0.4, 0.9$ , partly stuck regime with  $y = 1$  and  $\tau = 1$  and viscous regime with  $y = 3$  and  $\tau = 1$ .

The proof was done in a general case in [4]. For the sake of completeness we recall it. It is sufficient to show the case  $a \geq 0$  i.e.  $y \geq 0$ .

### 3. Proof

#### 3.1. Stuck regime

We observe that the potential  $U$  attains its minimum  $\frac{y^2}{2\tau}$  at  $v = 0$ . We have

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D})}{\nu} \leq r) = \frac{\int_{lv}^{r\nu} \exp(-\frac{U(v)}{\nu}) dv}{\int_{-\infty}^{+\infty} \exp(-\frac{U(v)}{\nu}) dv}.$$

Multiplying the denominator and the nominator by  $\exp(\frac{y^2}{2\tau\nu})$ , and using the change of variable  $\frac{v}{\nu}$  we have

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D})}{\nu} \leq r) = \frac{\int_l^r \exp \left[ -|v|(1 - \text{sgn}(v)\frac{y}{\tau}) - \sqrt{\nu}\frac{v^2}{2\tau} \right] dv}{\int_{-\infty}^{+\infty} \exp \left[ -|v|(1 - \text{sgn}(v)\frac{y}{\tau}) - \sqrt{\nu}\frac{v^2}{2\tau} \right] dv}.$$

The latter converges to

$$\frac{\int_l^r \exp \left[ -|v|(1 - \operatorname{sgn}(v)\frac{y}{\tau}) \right] dv}{\int_{-\infty}^{+\infty} \exp \left[ -|v|(1 - \operatorname{sgn}(v)\frac{y}{\tau}) \right] dv}$$

as  $\nu \rightarrow 0$ , which achieves the proof of the stuck regime.

### 3.2. Partly stuck regime

a) We are going to prove for each  $l < r \leq 0$  that

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D})}{\nu} \leq r \mid v(\frac{T}{D}) < 0) \rightarrow \int_l^r 2 \exp(2v) dv.$$

We have

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D})}{\nu} \leq r \mid v(\frac{T}{D}) < 0) = \frac{\int_{l\nu}^{r\nu} \exp \left[ -\frac{1}{\nu}(-v + \frac{(v-\tau)^2}{2\tau}) \right] dv}{\int_{-\infty}^0 \exp \left[ -\frac{1}{\nu}(|v| + \frac{(v-\tau)^2}{2\tau}) \right] dv}.$$

Multiplying the denominator and the nominator by  $\frac{\exp(\frac{\tau}{2\nu})}{\nu}$ , and using the change of variable  $\frac{v}{\nu}$  we obtain

$$\frac{\int_l^r \exp \left( 2v - \sqrt{\nu} \frac{v^2}{2t} \right) dv}{\int_{-\infty}^0 \exp \left( 2v - \sqrt{\nu} \frac{v^2}{2t} \right) dv}.$$

The latter converges to

$$\frac{\int_l^r \exp(2v) dv}{\int_{-\infty}^0 \exp(2v) dv}$$

as  $\nu \rightarrow 0$ , which achieves the proof of the part 1.

b) We have, for  $0 < l < r$ ,

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D})}{\sqrt{\nu}} \leq r \mid v(\frac{T}{D}) > 0) = \frac{\int_{l\sqrt{\nu}}^{r\sqrt{\nu}} \exp \left[ -\frac{1}{\nu}(v + \frac{(v-\tau)^2}{2\tau}) \right] dv}{\int_0^{+\infty} \exp \left[ -\frac{1}{\nu}(v + \frac{(v-\tau)^2}{2\tau}) \right] dv}.$$

Multiplying the denominator and the nominator by  $\frac{\exp(\frac{\tau}{2\nu})}{\sqrt{\nu}}$  and using the change of variable  $\frac{v}{\sqrt{\nu}}$  we get the proof of the first part of b).

For the second part we use the same proof and show that  $\mathbb{P}(av(\frac{T}{D}) > 0) \rightarrow 1$  as  $D \rightarrow 0$ .

### 3.3. Viscous regime

The main tool of the proof is the following well known result see e.g.[1].

**Lemma:** Let  $H$  be any measurable map such that

$$\int_{-\infty}^{+\infty} \exp(-H(v)) dv < +\infty$$

and

$$\inf \{ H(v) : |v - v_0| \geq \delta \} > H(v_0)$$

for some  $v_0$  and  $\delta > 0$ . Then for any  $\gamma > 0$ ,

$$\nu^{-\gamma} \int_{|v-v_0| \geq \delta} \exp \left[ -\frac{1}{\nu} (H(v) - H(v_0)) \right] dv \rightarrow 0$$

as  $\nu \rightarrow 0$ .

Now, let us apply this lemma with  $H(v) = U(v)$  and  $v_0 = y - \tau$  the minimizer of  $U$ . We have, for  $l < r$ ,

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D}) - (y - \tau)}{\sqrt{\nu}} \leq r) = \frac{\int_{l\sqrt{\nu}+y-\tau}^{r\sqrt{\nu}+y-\tau} \exp(-\frac{1}{\nu}U(v)) dv}{\int_{-\infty}^{+\infty} \exp(-\frac{1}{\nu}U(v)) dv}.$$

We have for  $v > 0$ , that

$$U(v) - U(y - \tau) = \frac{(v - (y - \tau))^2}{2\tau}.$$

If  $l > -\infty$ , then for small  $\nu$ , we have

$$\begin{aligned} \int_{l\sqrt{\nu}+y-\tau}^{r\sqrt{\nu}+y-\tau} \exp \left[ -\frac{1}{\nu} (U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} &= \int_{l\sqrt{\nu}+y-\tau}^{r\sqrt{\nu}+y-\tau} \exp \left[ -\frac{1}{2\tau\nu} (v - (y - \tau))^2 \right] \frac{dv}{\sqrt{\nu}} \\ &= \int_l^r \exp(-\frac{v^2}{2\tau}) dv, \end{aligned}$$

and then

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D}) - (y - \tau)}{\sqrt{\nu}} \leq r) = \int_l^r \exp(-\frac{v^2}{2\tau}) \frac{dv}{\sqrt{2\pi\tau}}.$$

If  $l = -\infty$ , then

$$\int_{-\infty}^{r\sqrt{\nu}+y-\tau} \exp \left[ -\frac{1}{\nu} (U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} = (1) + (2),$$

where

$$\begin{aligned} (1) &= \int_{[v < 0]} \exp \left[ -\frac{1}{\nu} (U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} \rightarrow 0, \\ (2) &= \int_{[0 \leq v \leq r\sqrt{\nu}+y-\tau]} \exp \left[ -\frac{1}{\nu} (U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}}. \end{aligned}$$

From Lemma (3.3) the term (1) converges to 0. By the change of variable  $z = \frac{v-(y-\tau)}{\sqrt{\nu}}$ , the term (2)

$$(2) = \int_{[-\frac{(y-\tau)}{\sqrt{\nu}} \leq z \leq r]} \exp(-\frac{z^2}{2\tau}) dz$$

converges to  $\int_{-\infty}^r \exp(-\frac{z^2}{2\tau}) dz$ . By taking  $r = +\infty$ , we get

$$\int_{-\infty}^{+\infty} \exp \left[ -\frac{1}{\nu} (U(v) - U(y - \tau)) \right] \frac{dv}{\sqrt{\nu}} \rightarrow \int_{-\infty}^{+\infty} \exp(-\frac{z^2}{2\tau}) dz,$$

and then

$$\mathbb{P}(l \leq \frac{v(\frac{T}{D}) - (y - \tau)}{\sqrt{\nu}} \leq r) \rightarrow \int_l^r \exp(-\frac{v^2}{2\tau}) \frac{dv}{\sqrt{2\pi\tau}},$$

which achieves the proof.

#### 4. Time-dependent propagator

Now we drop the coefficient  $\frac{1}{2}$  in (1) and we discuss the calculation of the time-dependent propagator of

$$dv = -[\alpha v + a + \Delta \operatorname{sgn}(v)]dt + \sqrt{D}dB.$$

Using the equality of the laws or the probability distributions of  $(\sqrt{D}B(\frac{t}{D}))$  and  $(B(t))$ , we derive that

$$\operatorname{Law}(v^{\alpha,a,\Delta,D}(t)) = \operatorname{Law}(v^{\frac{\alpha}{D},\frac{a}{D},\frac{\Delta}{D},1}(Dt)).$$

Hence the propagators  $p^{\alpha,a,\Delta,D}(v, t | v_0, 0)$ ,  $p^{\frac{\alpha}{D},\frac{a}{D},\frac{\Delta}{D},1}(v, t | v_0, 0)$  respectively of  $v^{\alpha,a,\Delta,D}(t)$  and  $v^{\frac{\alpha}{D},\frac{a}{D},\frac{\Delta}{D},1}(t)$  satisfy the relation

$$p^{\alpha,a,\Delta,D}(v, t | v_0, 0) = p^{\frac{\alpha}{D},\frac{a}{D},\frac{\Delta}{D},1}(v, Dt | v_0, 0).$$

Hence, it is sufficient to study the case  $D = 1$ .

#### 5. Time-dependent propagator for $\alpha = a = 0$ using local occupation time

We denote by  $\mathbb{P}$  and  $\mathbb{P}_{v_0}$  the probability distribution respectively of the trajectories  $s \in [0, t] \rightarrow v(s)$  of the solution of (1) and the Brownian motion starting from  $v_0$ .

Under the probability distribution

$$\exp\left(-\Delta \int_0^t \operatorname{sgn}(B_s)dB_s - \frac{t\Delta^2}{2}\right) d\mathbb{P}_{v_0} := f_{\operatorname{sgn}(B)}d\mathbb{P}_{v_0}$$

the process  $(B(s) : s \in [0, t])$  is solution of the equation

$$dv = -\Delta \operatorname{sgn}(v)dt + dB, \quad v(0) = v_0. \quad (3)$$

We simplify the stochastic integral  $\int_0^t \operatorname{sgn}(B_s)dB_s$  using Tanaka formula [6]

$$|B_t| = |v_0| + \int_0^t \operatorname{sgn}(B_s)dB_s + 2L_t.$$

Here the local time

$$\begin{aligned} L_t &= \lim_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon} \int_0^t \mathbf{1}_{[|B_s| \leq \varepsilon]} ds \\ &= \frac{1}{2} \int_0^t \delta(B_s) ds. \end{aligned}$$

It follows that

$$-\int_0^t \operatorname{sgn}(B_s)dB_s = |v_0| - |B_t| + 2L_t.$$

Now,

$$f_{\operatorname{sgn}(B)}(B) = \exp\left(\Delta(|v_0| - |B_t| + 2L_t) - \frac{t\Delta^2}{2}\right).$$

The densities of  $v(t)$  and the Brownian motion  $B(t)$  are related by

$$p(v, t | v_0) = \mathbb{E}_{v_0} \left[ \delta(B_t - v) \exp\left(\Delta(|v_0| - |B_t| + 2L_t) - \frac{t\Delta^2}{2}\right) \right].$$

The latter formula is also known as path integral representation [2]. Hence the law of the solution  $v(t)$  is given by the law of  $(B_t, L_t)$ .

### 5.1. Density of Brownian motion and its local time

Set  $\Gamma_t = \int_0^t \mathbf{1}[B_s \geq 0] ds$ , and

$$h(s, v) = \frac{|v|}{\sqrt{2s^3\pi}} \exp\left(-\frac{v^2}{2s}\right), \quad s > 0, v \in \mathbb{R}.$$

Karatzas and Shreve [5] have calculated the probability density  $\mathbb{P}_{v_0}(B_t \in db, L_t \in dl, \Gamma_t \in d\tau) := p_t(dv, dl, d\tau | v_0)$  of  $(B_t, L_t, \Gamma_t)$  as follows. For  $v_0 \geq 0$  we have

$$\begin{aligned} \mathbb{P}_{v_0}(B_t \in db, L_t \in dl, \Gamma_t \in d\tau) &= 2h(\tau, l + v_0)h(t - \tau, l - b)dbdl, \quad b < 0, \\ \mathbb{P}_{v_0}(B_t \in db, L_t \in dl, \Gamma_t \in d\tau) &= 2h(t - \tau, l)h(\tau, l + b + v_0)dbdl, \quad b > 0, \\ \mathbb{P}_{v_0}(B_t \in db, L_t = 0, \Gamma_t = t) &= \omega(v_0, b, t), \quad b > 0, v_0 \geq 0, \end{aligned}$$

where

$$\begin{aligned} \omega(v_0, b, t) &= \gamma_t(b - v_0) - \gamma_t(b + v_0), \\ \gamma_t(u) &= \frac{1}{\sqrt{2t\pi}} \exp\left(-\frac{u^2}{2t}\right). \end{aligned}$$

We derive the joint distribution of  $(B_t, L_t)$  under  $\mathbb{P}_{v_0}$  with  $v_0 \geq 0$ :

$$\begin{aligned} \mathbb{P}_{v_0}(B_t \in db, L_t \in dl) &= 2 \frac{(2l + v_0 - b)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2l + v_0 - b)^2}{2t}\right] dbdl, \quad b < 0, l > 0, \\ \mathbb{P}_{v_0}(B_t \in db, L_t \in dl) &= 2 \frac{(2l + v_0 + b)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2l + v_0 + b)^2}{2t}\right] dbdl + \omega(v_0, b, t)\delta(l), \quad b > 0, l \geq 0. \end{aligned}$$

Now, we calculate the density of the solution (3) as follows. If  $v < 0$ , then

$$\begin{aligned} p(v, t | v_0) &= \mathbb{E}_{v_0} \left[ \delta(B(t) - v) \exp\left(\Delta(v_0 - |B_t| + 2L_t) - \frac{t\Delta^2}{2}\right) \right] \\ &:= \exp\left(\Delta(v_0 + v) - \frac{t\Delta^2}{2}\right) \mathbb{E}_{v_0} [\delta(B(t) - v) \exp(2\Delta L_t)] \\ &= 2 \exp\left(\Delta(v_0 + v) - \frac{t\Delta^2}{2}\right) \int_0^{+\infty} \frac{(2l - v + v_0)}{\sqrt{2t^3\pi}} \exp(2\Delta l - \frac{(2l - v + v_0)^2}{2t}) dl \\ &= \exp(\Delta(v_0 + v) - \frac{t\Delta^2}{2}) \int_0^{+\infty} \frac{(l - v + v_0)}{\sqrt{2t^3\pi}} \exp(\Delta l - \frac{(l - v + v_0)^2}{2t}) dl \\ &= \exp(\Delta(v_0 + v) - \frac{t\Delta^2}{2}) \left[ \frac{1}{\sqrt{2t\pi}} \exp\left(-\frac{(v_0 - v)^2}{2t}\right) + \Delta \int_0^{+\infty} \exp(\Delta l - \frac{(l - v + v_0)^2}{2t}) \frac{dl}{\sqrt{2t\pi}} \right] \\ &= \frac{1}{\sqrt{2t\pi}} \exp\left(-\frac{t\Delta^2}{2}\right) \exp(\Delta(v_0 + v)) \exp\left(-\frac{(v_0 - v)^2}{2t}\right) + \\ &\quad \Delta \exp(\Delta(v_0 + v) - \frac{t\Delta^2}{2}) \int_0^{+\infty} \exp(\Delta l - \frac{(l + v_0 - v)^2}{2t}) \frac{dl}{\sqrt{2t\pi}}. \end{aligned}$$

After some calculation we obtain

$$\int_0^{+\infty} \exp(\Delta l - \frac{(l + v_0 - v)^2}{2t}) \frac{dl}{\sqrt{2t\pi}} = \exp\left(\frac{\Delta^2 t}{2}\right) \exp(\Delta(v - v_0)) F\left(\frac{v - v_0 + \Delta t}{\sqrt{t}}\right),$$

where  $F(v) = \int_{-\infty}^v \frac{\exp(-\frac{u^2}{2})}{\sqrt{2\pi}} du$ . Finally for  $v_0 \geq 0$ ,  $v < 0$ , we have

$$p(v, t | v_0) = \left( \exp\left(-\frac{t\Delta^2}{2}\right) \gamma_t(v_0 - v) \exp(\Delta(v_0 - v)) + F\left(\frac{v - v_0 + \Delta t}{\sqrt{t}}\right) \right) \Delta \exp(2\Delta v).$$

If  $v > 0$ , then

$$\begin{aligned}
p(v, t | v_0) &= \mathbb{E}_{v_0} \left[ \delta(B(t) - v) \exp \left( \Delta(v_0 - |B_t| + 2L_t) - \frac{t\Delta^2}{2} \right) \right] \\
&:= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \mathbb{E}_{v_0} \left[ \delta(B(t) - v) \exp(2\Delta L_t) \right] \\
&= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \left[ 2 \int_0^{+\infty} \frac{(2l + v + v_0)}{\sqrt{2t^3\pi}} \exp(2\Delta l - \frac{(2l + v + v_0)^2}{2t}) dl + \omega(v_0, v, t) \right] \\
&= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \omega(v_0, v, t) \\
&\quad + \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \int_0^{+\infty} \frac{(l + v + v_0)}{\sqrt{2t^3\pi}} \exp(\Delta l - \frac{(l + v + v_0)^2}{2t}) dl \\
&= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \omega(v_0, v, t) + \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \left[ \frac{1}{\sqrt{2t\pi}} \exp \left( -\frac{(v_0 + v)^2}{2t} \right) \right. \\
&\quad \left. + \Delta \int_0^{+\infty} \exp \left( \Delta l - \frac{(l + v + v_0)^2}{2t} \right) \frac{dl}{\sqrt{2t\pi}} \right] \\
&= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \omega(v_0, v, t) + \frac{1}{\sqrt{2t\pi}} \exp \left( -\frac{t\Delta^2}{2} \right) \exp(\Delta(v_0 - v)) \exp \left( -\frac{(v_0 + v)^2}{2t} \right) \\
&\quad + \Delta \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \int_0^{+\infty} \exp \left( \Delta l - \frac{(l + v_0 + v)^2}{2t} \right) \frac{dl}{\sqrt{2t\pi}}.
\end{aligned}$$

From some calculation we obtain

$$\int_0^{+\infty} \exp \left( \Delta l - \frac{(l + v_0 + v)^2}{2t} \right) \frac{dl}{\sqrt{2t\pi}} = \exp \left( \frac{\Delta^2 t}{2} \right) \exp(-\Delta(v + v_0)) F \left( \frac{\Delta t - (v + v_0)}{\sqrt{t}} \right).$$

Finally if  $v > 0$ ,  $v_0 \geq 0$ , then

$$\begin{aligned}
p(v, t | v_0) &= \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \omega(v_0, v, t) + \frac{1}{\sqrt{2t\pi}} \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \exp \left( -\frac{(v_0 + v)^2}{2t} \right) \\
&\quad + \Delta \exp(-2\Delta v) F \left( \frac{\Delta t - (v + v_0)}{\sqrt{t}} \right) \\
&= \frac{1}{\sqrt{2t\pi}} \exp \left( \Delta(v_0 - v) - \frac{t\Delta^2}{2} \right) \exp \left( -\frac{(v_0 - v)^2}{2t} \right) + \Delta \exp(-2\Delta v) F \left( \frac{\Delta t - (v + v_0)}{\sqrt{t}} \right) \\
&= \left( \exp \left( \Delta(v_0 + v) - \frac{t\Delta^2}{2} \right) \gamma_t(v - v_0) + F \left( \frac{\Delta t - (v + v_0)}{\sqrt{t}} \right) \right) \Delta \exp(-2\Delta v).
\end{aligned}$$

Finally we have for  $v, v_0 \in \mathbb{R}$ , that

$$p(v, t | v_0) = q(v, t | v_0) \exp(-2\Delta|v|)$$

where

$$q(v, t | v_0) = \Delta \left( \exp(\Delta(|v_0| + |v|) - \frac{t\Delta^2}{2}) \gamma_t(v - v_0) + F \left( \frac{\Delta t - (|v| + |v_0|)}{\sqrt{t}} \right) \right).$$

Observe that  $q(v, t | v_0)$  is symmetric, i.e.  $q(v, t | v_0) = q^\Delta(v_0, t | v)$ . In the language of linear diffusion  $m(v) = \exp(-2\Delta|v|)$  is the speed measure of the linear diffusion (3).



## 6. The case $a \neq 0$

In this case the probability distribution  $\mathbb{P}$  of the solution

$$dv = -[\Delta \operatorname{sgn}(v) + a]dt + dB, \quad v(0) = v_0,$$

is also absolutely continuous with respect to  $\mathbb{P}_{v_0}$  (the probability distribution of the Brownian motion starting from  $v_0$ ). We have

$$\frac{d\mathbb{P}}{d\mathbb{P}_{v_0}}(B) = \exp\left(-\int_0^t (\Delta \operatorname{sgn}(B_s) + a)dB_s - \frac{1}{2} \int_0^t (\Delta \operatorname{sgn}(B_s) + a)^2 ds\right).$$

After some calculation we have

$$\begin{aligned} -\int_0^t (\Delta \operatorname{sgn}(B_s) + a)dB_s &= \Delta(|v_0| - |B_t| + a(v_0 - B_t)) + 2\Delta L_t, \\ \int_0^t (\Delta \operatorname{sgn}(B_s) + a)^2 ds &= (\Delta^2 + a^2)t + 2a\Delta(2\Gamma_t - t). \end{aligned}$$

It follows that

$$p(v, t | v_0) = \exp\left[\Delta(|v_0| - |v| + a(v_0 - v)) - \frac{(\Delta - a)^2 t}{2}\right] \mathbb{E}_{v_0}[\delta(B_t - v) \exp(2\Delta L_t - 2a\Delta\Gamma_t)].$$

Then  $p(v, t | v_0)$  is calculated using the trivariate probability distribution  $p_t(db, dl, d\tau)$  of  $(B_t, L_t, \Gamma_t)$  as follows:

$$p(v, t | v_0) = \exp\left[\Delta(|v_0| - |v| + a(v_0 - v)) - \frac{(\Delta - a)^2 t}{2}\right] \int_0^{+\infty} \int_0^t \exp(2\Delta(l - a\tau)) p_t(v, dl, d\tau).$$

## 7. The general case

Similarly as above the density of the solution of

$$dv = -[\alpha v + \Delta \operatorname{sgn}(v) + a]dt + dB, \quad v(0) = v_0,$$

is

$$\begin{aligned} p(v, t | v_0) &= \exp\left[\Delta(|v_0| - |v| + a(v_0 - v)) - \frac{(\Delta - a)^2 t}{2} + \frac{\alpha t}{2}\right] \\ &\int_0^{+\infty} \int_0^t \int_0^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} \exp(2\Delta l - 2a\Delta\tau - \frac{\alpha^2}{2}b_2 - \alpha\Delta|b_1| - a\alpha b_1) p_t(v, dl, d\tau, db_1, d|b_1|, db_2), \end{aligned}$$

where  $p_t(db, dl, d\tau, db_1, d|b_1|, db_2)$  is the probability density of

$$(B_t, L_t, \Gamma_t, \int_0^t B_s ds, \int_0^t |B_s| ds, \int_0^t B_s^2 ds).$$

## 8. Conclusion

We have precised and extended the three different regimes of the Langevin equation which includes a viscous friction force, a Coulombic friction and a constant external force. Moreover we find again its time-dependent propagator using the density of Brownian motion, its local and occupation times.

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